Name ____

Student ID ____________________ TA/Section (circle):

Katie	12pm-C01	1pm-C02	2pm-C03
Mary	3pm-C04	8pm-C05	9pm-C06

Math 20F, Winter 2010, Midterm Exam 2 Solutions

- ***Show all of your work and justify your answers to receive full credit.***
- *Write your answers and work clearly and legibly; no credit will be given for illegible solutions.*
- *Go back and check your answers if you finish early.*

#	Points	Score
1	6	
2	3	
3	3	
4	2	
5	2	
6	2	
7	2	
Σ	20	

1. (6 points) Let

$$A = \begin{bmatrix} 1 & -1 & 2 & 3 & 0 \\ 2 & -1 & 4 & 11 & 3 \\ -1 & 3 & -2 & 8 & 4 \\ 1 & 1 & 2 & 14 & 4 \end{bmatrix}$$

(a) Find a basis for Col A.
(b) Find a basis for Row A.
(c) Find a basis for Nul A.
Answer: We need to row reduce to echelon form to determine a basis for Col A and Row A, and we need reduced echelon form to determine a basis for Nul A.

$$\begin{bmatrix} 1 & -1 & 2 & 3 & 0 \\ 2 & -1 & 4 & 11 & 3 \\ -1 & 3 & -2 & 8 & 4 \\ 1 & 1 & 2 & 14 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 & 3 & 0 \\ 0 & 1 & 0 & 5 & 3 \\ 0 & 2 & 0 & 11 & 4 \\ 0 & 2 & 0 & 11 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 8 & 3 \\ 0 & 1 & 0 & 5 & 3 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 8 & 3 \\ 0 & 1 & 0 & 5 & 3 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & 2 & 0 & 19 \\ 0 & \textcircled{1} & 0 & 0 & 13 \\ 0 & 0 & 0 & \textcircled{1} & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The pivots are circled. The pivot columns are columns 1, 2 and 4. The pivot columns of matrix A form a basis for Col A, so

$$\text{Basis for Col}\, A = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 11 \\ 8 \\ 14 \end{bmatrix} \right\}$$

The nonzero rows of an echelon form of A form a basis for Row A, so

$$\text{Basis for Row}\, A = \{(1,0,2,0,19), (0,1,0,0,13), (0,0,0,1,-2)\}$$

To find the null space we look at

$$\begin{aligned} x_1 \quad + 2x_3 \quad\quad + 19x_5 &= 0 \\ x_2 \quad\quad\quad + 13x_5 &= 0 \\ + x_4 - \ 2x_5 &= 0 \end{aligned}$$

where x_3 and x_5 are free variables which leads to

$$\begin{aligned} x_1 &= -2x_3 - 19x_5 \\ x_2 &= -13x_5 \\ x_3 &= x_3 \\ x_4 &= 2x_5 \\ x_5 &= x_5 \end{aligned} \quad \Longrightarrow \quad \mathbf{x} = \begin{bmatrix} -2x_3 \\ 0 \\ x_3 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -19x_5 \\ -13x_5 \\ 0 \\ 2x_5 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -19 \\ -13 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

Therefore,

$$\text{Basis for Nul}\, A = \left\{ \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -19 \\ -13 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}$$

2. (3 points) Find the determinant of the matrix

$$\begin{bmatrix} 2 & 3 & -2 & 1 \\ 0 & 2 & 5 & 4 \\ 0 & -3 & 2 & -3 \\ 0 & 1 & 1 & 2 \end{bmatrix}$$

Answer: Cofactor expansion down the first column saves work. After that, there are various ways to complete the computation.

$$\begin{vmatrix} 2 & 3 & -2 & 1 \\ 0 & 2 & 5 & 4 \\ 0 & -3 & 2 & -3 \\ 0 & 1 & 1 & 2 \end{vmatrix} = 2\begin{vmatrix} 2 & 5 & 4 \\ -3 & 2 & -3 \\ 1 & 1 & 2 \end{vmatrix} = 2\begin{vmatrix} 0 & 3 & 0 \\ 0 & 5 & 3 \\ 1 & 1 & 2 \end{vmatrix} = 2 \cdot 1 \cdot \begin{vmatrix} 3 & 0 \\ 5 & 3 \end{vmatrix} = 2 \cdot 1 \cdot 9 = 18$$

The 3×3 matrix was manipulated by adding -2 times the third row to the first row, and adding 3 times the third row to the second row. These row operations do not change the determinant. The 3×3 matrix was then expanded down the first column.

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3. (3 points) The matrix

$$A = \begin{bmatrix} 4 & 2 & 3 \\ 0 & 4 & 1 \\ 2 & 3 & 8 \end{bmatrix}$$

has $\lambda = 3$ as an eigenvalue. Find an eigenvector corresponding to this eigenvalue.

Answer: An eigenvector will be any nonzero vector in $\mathrm{Nul}(A - 3I)$. That is, a nonzero solution to $(A - 3I)\mathbf{x} = \mathbf{0}$.

$$A - 3I = \begin{bmatrix} 4 & 2 & 3 \\ 0 & 4 & 1 \\ 2 & 3 & 8 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 2 & 3 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

This implies

$$\begin{aligned} x_1 \quad + x_3 &= 0 \\ x_2 + x_3 &= 0 \\ x_3 \text{ is free} & \end{aligned} \quad \Longrightarrow \quad \begin{aligned} x_1 &= -x_3 \\ x_2 &= -x_3 \\ x_3 &= x_3 \end{aligned} \quad \Longrightarrow \quad \mathbf{x} = \begin{bmatrix} -x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

So the eigenspace corresponding to $\lambda = 3$ is Span $\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$, and any nonzero vector in this set is an eigenvector. In particular, $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ is an eigenvector.

4. (2 points) Let

$$A = \begin{bmatrix} 1 & 5 & -1 \\ 3 & 7 & -11 \\ -2 & -2 & 10 \end{bmatrix}$$

Is the vector $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ in the column space of A? Justify your answer.

Answer:

The column space of A is defined as the span of the columns of A, which (by definition of span) is all linear combinations of the columns of A. So we need to determine if

$$x_1 \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 5 \\ 7 \\ -2 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -11 \\ 10 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

for some real numbers x_1, x_2 and x_3. This is the same as asking if

$$\begin{bmatrix} 1 & 5 & -1 \\ 3 & 7 & -11 \\ -2 & -2 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

has a solution. By row reducing the augmented matrix,

$$\left[\begin{array}{ccc|c} 1 & 5 & -1 & 1 \\ 3 & 7 & -11 & 1 \\ -2 & -2 & 10 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 5 & -1 & 1 \\ 0 & -8 & -8 & -2 \\ 0 & 8 & 8 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 5 & -1 & 1 \\ 0 & -8 & -8 & -2 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

we see that the system is inconsistent (does not have a solution). Therefore, the vector is not in $\operatorname{Col} A$.

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5. (2 points)

(a) If A is a 4×3 matrix, what is the largest possible dimension of the row space of A?

(b) If A is a 3×4 matrix, what is the largest possible dimension of the row space of A?

Answer:

(a) If A is 4×3, then there are four rows, and the rows are vectors in $\mathbb{R}^3$, so

$$\operatorname{Row} A = \operatorname{Span}\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4\}$$

is a subspace of $\mathbb{R}^3$. Therefore, the maximum dimension that $\operatorname{Row} A$ can be is three.

(b) If A is 3×4, then the rows are vectors in $\mathbb{R}^4$, but there are only three rows, so the dimension of

$$\operatorname{Row} A = \operatorname{Span}\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\}$$

can not be more than three. (A span of p vectors can be at most p-dimensional.)

A more general argument shows that for an $m \times n$ matrix,

$$\operatorname{Rank} A = \dim \operatorname{Col} A = \dim \operatorname{Row} A \leq \min(m, n)$$

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6. (2 points) Let A be an $m \times n$ matrix. Show that the null space of A,

$$\text{Nul}\, A = \{\mathbf{x} \colon \mathbf{x} \text{ in } \mathbb{R}^n \text{ and } A\mathbf{x} = \mathbf{0}\},$$

is closed under vector addition. (This should only take a few lines.)
Answer: Let $\mathbf{u}$ and $\mathbf{v}$ be in $\text{Nul}\, A$. By the definition of null space, this means

$$A\mathbf{u} = \mathbf{0} \quad \text{and} \quad A\mathbf{v} = \mathbf{0}$$

Now,

$$A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = \mathbf{0} + \mathbf{0} = \mathbf{0}$$

so $\mathbf{u} + \mathbf{v}$ satisfies the definition of null space, so $\mathbf{u} + \mathbf{v}$ is in $\text{Nul}\, A$. Therefore, $\text{Nul}\, A$ is closed under vector addition.

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7. (2 points) Suppose A is $n \times n$, and for some $\mathbf{b}$ in $\mathbb{R}^n$ the equation $A\mathbf{x} = \mathbf{b}$ has more than one solution. Can the columns of A span $\mathbb{R}^n$. Why or why not? Explain.
Answer:
The idea is to show that one statement of the Invertible Matrix Theorem (IMT) is false, then we know that they are all false, in particular that the columns of A do not span $\mathbb{R}^n$

Argument 1: If $A\mathbf{x} = \mathbf{b}$ has more than one solution, then the mapping $\mathbf{x} \mapsto A\mathbf{x}$ is not one-to-one (by definition of one-to-one). It follows by the IMT that the columns of A do not span $\mathbb{R}^n$.

Argument 2: The system $A\mathbf{x} = \mathbf{0}$ has nontrivial solutions: If $\mathbf{u} \neq \mathbf{v}$ and $A\mathbf{u} = \mathbf{b}$ and $A\mathbf{v} = \mathbf{b}$, then

$$A(\underbrace{\mathbf{u} - \mathbf{v}}_{\neq \mathbf{0}}) = A\mathbf{u} - A\mathbf{v} = \mathbf{b} - \mathbf{b} = \mathbf{0}.$$

Now, by the IMT, the columns of A do not span $\mathbb{R}^n$.

Argument 3: This does not use the IMT. The system $A\mathbf{x} = \mathbf{b}$ must have at least one free variable. This means that there are less than n pivot columns which implies that a basis for $\text{Col}\, A$ contains less than n vectors, so $\dim \text{Col}\, A < n$. Therefore, $\text{Col}\, A \neq \mathbb{R}^n$ because $\dim \mathbb{R}^n = n$.