Math 20F Practice Midterm 2.

Problem 1. True or False: For each statement below, determine whether it is true or false, and circle the appropriate letter. You do not need to justify your answer.

(**T** **F**) If A is a (4×6) matrix and dim(Nul(A))= 2, then $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b} \in \mathbb{R}^4$.
Solution: True. By the Rank theorem on pg. 265, if dim(Nul(A))= 2, then dim(Col(A))= 4, so the span of the columns of A is four-dimensional, and thus is all of $\mathbb{R}^4$.

(**T** **F**) The set $\{\begin{bmatrix} x \\ y \end{bmatrix} | xy \geq 0\}$ is a subspace of $\mathbb{R}^2$.

Solution: False. Clearly the vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$ are in the set. However, their sum is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, which is not in the set. Thus it is not closed under vector addition, and not a subspace.

(**T** **F**) If A is a square matrix and $\det A = 0$, then 0 is an eigenvalue for $A^T A$.
Solution: True. By the Invertible Matrix Theorem on pg. 312, $\det A^T A = 0$ if and only if 0 is an eigenvalue of $A^T A$. But $\det A^T A = (\det A^T)(\det A) = 0$.

(**T** **F**) Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ be vectors in a vector space V, and let $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$. If $\dim H = 3$, and $\mathbf{v}_1 - \mathbf{v}_3 = \mathbf{v}_4$, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for H.
Solution: True. By the Spanning Set Theorem on pg. 239, the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ spans H, since $\mathbf{v}_4$ is a linear combination of the other vectors. Further, by the Basis Theorem on pg. 259, any set with three elements that spans a three-dimensional space is a basis.

Problem 2. Let $A = \begin{bmatrix} -2 & 12 \\ -1 & 5 \end{bmatrix}$.

a) Find the eigenvalues of A.

Solution: The eigenvalues are the solutions to $\det(A - \lambda I) = 0$. $\det(A - \lambda I) = \det \begin{bmatrix} -2-\lambda & 12 \\ -1 & 5-\lambda \end{bmatrix} = (-2-\lambda)(5-\lambda) - 12 = \lambda^2 - 3\lambda + 2$. The roots are $\lambda = 1$ and $\lambda = 2$, so these are the eigenvalues of A.

b) Find a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$.

Solution: We must find an eigenvector for each eigenvalue. To do this, we find a nonzero vector in each of the null spaces of $A - I$ and $A - 2I$.

$A - 2I = \begin{bmatrix} -4 & 12 \\ -1 & 3 \end{bmatrix}$, which row reduces to $\begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$, so any solution $\mathbf{x}$ to $(A - 2I)\mathbf{x} = \mathbf{0}$ satisfies $x_1 = 3x_2$, and $\mathbf{x} = \begin{bmatrix} 3x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Thus $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = 2$.

Similarly, $A - I = \begin{bmatrix} -3 & 12 \\ -1 & 4 \end{bmatrix}$, which row reduces to $\begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix}$, so any solution $\mathbf{x}$ to $(A - I)\mathbf{x} = \mathbf{0}$ satisfies $x_1 = 4x_2$, and $\mathbf{x} = \begin{bmatrix} 4x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 4 \\ 1 \end{bmatrix}$. Thus $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = 1$.

Using these eigenvectors as the columns of P, and using the eigenvalues as the entries on the diagonal of D, we find:

$P = \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix}$, and $D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$.

Problem 3. a) Find all values for k such that $\det A = 0$, where $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & k \\ 1 & k & 3 \end{bmatrix}$

Solution: $\det A = 9 + k - 2k + 3 - k^2 - 6 = -k^2 - k + 6 = -(k+3)(k-2)$, so $\det A = 0$ if and only if $k = -3$ or $k = 2$.

b) Find all values for k such that the following system of equations has more than one solution:

$$\begin{aligned} x_1 + x_2 - x_3 &= 0 \\ 2x_1 + 3x_2 + kx_4 &= 0 \\ x_1 + kx_2 + 3x_3 &= 0 \end{aligned}$$

Solution: Note that this system is equivalent to the matrix equation $A\mathbf{x} = \mathbf{0}$ with A as given above. By the invertible matrix theorem, the equation $A\mathbf{x} = \mathbf{0}$ has more than one solution if and only if $\det A = 0$, which, by part (a), occurs exactly when $k = -3$ or $k = 2$.

Problem 4. Let V be the subspace of $C([0,1])$ spanned by $\sin t, \cos t, \sin^2 t$, and $\cos^2 t$. You may assume that this set of vectors is linearly independent.

Use coordinates to show that the set $\{1, 2\sin t + \sin^2 t, 3\cos t - \cos^2 t\}$ is linearly independent.

Hint: You need to use a simple trig identity to show that 1 is in V.

Solution: First note that $1 = \sin^2 t + \cos^2 t$. Let $B = \{\sin t, \cos t, \sin^2 t, \cos^2 t\}$. Because they are linearly independent, they form a basis for their span. Taking coordinates of the given vectors gives us:

$$[1]_B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \; [2\sin t + \sin^2 t]_B = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \text{ and } [3\cos t - \cos^2 t]_B = \begin{bmatrix} 0 \\ 3 \\ 0 \\ -1 \end{bmatrix}.$$

The given vectors are linearly independent if their coordinate vectors are linearly independent in $\mathbb{R}^4$, so we simply arrange them into a matrix and row reduce:

$$\begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

This matrix has a pivot in every column, and thus the columns are linearly independent.

Problem 5. Let $A = \begin{bmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{bmatrix}$, $B = \begin{bmatrix} \pi & \sqrt{2} \\ 0 & \ln 3 \\ 1 & 8 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 4 \\ 0 & 2 \end{bmatrix}$.

a) Find bases for Col(A) and Col(C).

Solution: First we find a basis for Col(A) by row reducing A to $\begin{bmatrix} 1 & -2 & 1 & 1 \\ 0 & -2 & -5 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. This matrix has pivots in columns 1, and 2, so columns 1, 2 of A form a basis for its column space. Thus a basis for Col(A) is

$$\left\{ \begin{bmatrix} -2 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \\ 8 \end{bmatrix} \right\}.$$

For C, the matrix is already row reduced and has a pivot in every column, so Col(C)$= \mathbb{R}^2$.

b) Let X be the block matrix defined by $X = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$.

Find a basis for Col(X). Solution: Because of the 0 in the lower-left corner of X, the matrix B has little to do with the column space. If we were to row reduce X, we would find, just as when we row reduced A, that columns 1, 2, and 4 have pivots, and column 3 does not. Further, the last two columns will also have pivots because C has pivots in each column. So a basis for the column space of X comprises all of its columns except the third, and we get

$$\left\{ \begin{bmatrix} -2 \\ 2 \\ -3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ -6 \\ 8 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \pi \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \sqrt{2} \\ \ln 3 \\ 8 \\ 4 \\ 2 \end{bmatrix} \right\}.$$