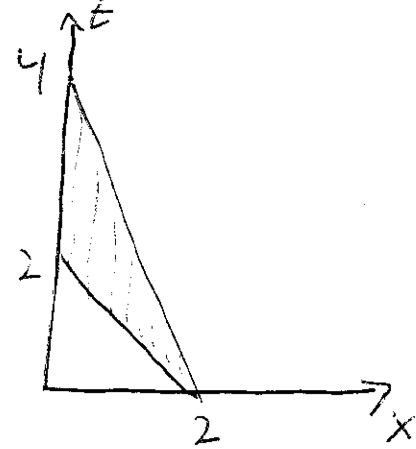
Problem 1. Find the mass of the solid body with the density function $\rho(x, y, z) = c \cdot y$ bounded by the surfaces $2x + z = 4$, $x + z = 2$, $y^2 = 2x$ and $y = 0$.

Region projects onto



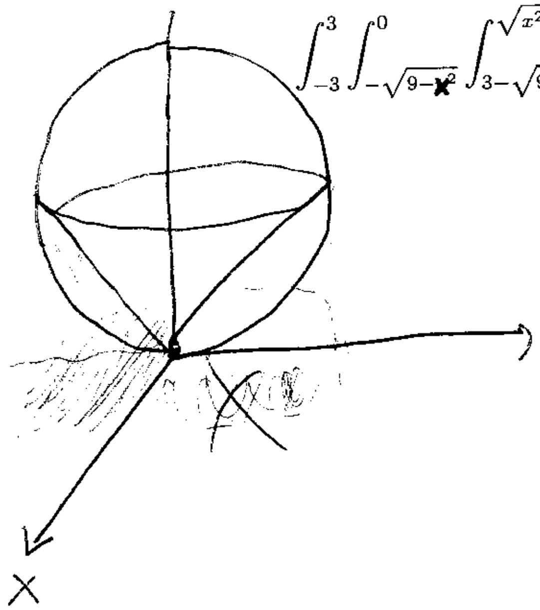
$$\int_0^2 \int_{2-x}^{4-2x} \int_0^{\sqrt{2x}} cy \, dy \, dz \, dx$$

$$= c \int_0^2 (2x - x^2) \, dx$$

$$= \frac{4c}{3}$$

Problem 2. Describe the region of integration and evaluate the integral by changing to spherical coordinates:

$$\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{0}\int_{3-\sqrt{9-(x^2+y^2)}}^{\sqrt{x^2+y^2}}(x^2+y^2)^2\,dzdydx.$$



It's the region inside the sphere with center (0,0,3) + radius 3 outside the right circular cone $z=\sqrt{x^2+y^2}$

Also possible:

Spherical

$$\int_{\pi/4}^{\pi/2}\int_{\pi}^{2\pi}\int_{0}^{6\cos\phi}\rho^6\sin^5\phi\,d\rho\,d\theta\,d\phi$$

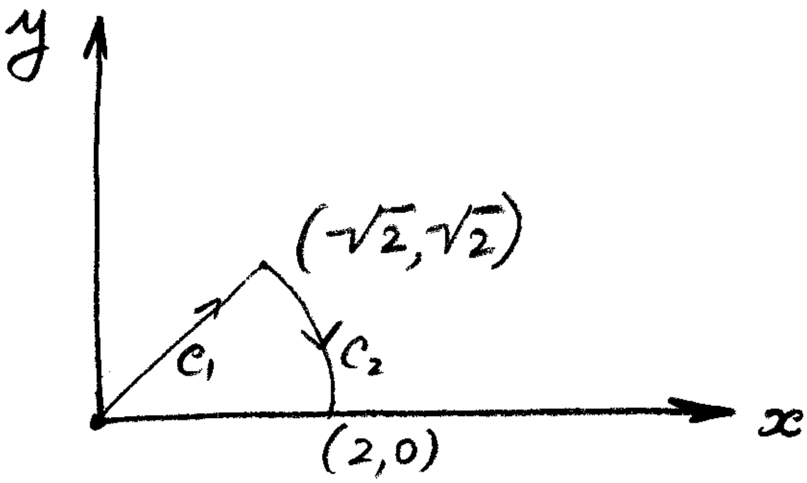
Cylindrical

$$\int_{\pi}^{2\pi}\int_{0}^{3}\int_{3-\sqrt{9-r^2}}^{r} r^5\,dz\,dr\,d\theta$$

$$= \pi\int_0^3\left(r^6-r^5+r^5\sqrt{9-r^2}\right)dr \rightarrow \int_0^3 r^5\sqrt{9-r^2}\,dr = \frac{1}{2}\int_0^9 (9-u)^2u^{1/2}\,du$$

$$= \frac{\pi 3^7}{7} - \frac{\pi 3^5}{5} + \pi 3^6 + \frac{2\cdot 3^7\pi}{5} + \frac{3^6\pi}{7}$$

Problem 3. Compute the integral $\int_C e^{\sqrt{x^2+y^2}} ds$, where C is the curve consisting of the segment of the straight line connecting $(0,0)$ and $(\sqrt{2},\sqrt{2})$, followed by the arc of the unit circle connecting the points $(\sqrt{2},\sqrt{2})$ and $(2,0)$.



C_1 : $r(t) = (\sqrt{2}t, \sqrt{2}t)$, $0 \le t \le 1$

$r'(t) = (\sqrt{2}, \sqrt{2})$

C_2 : $r(t) = (2\cos t, 2\sin t)$, $0 \le t \le \pi/4$, $r'(t) = 2(-\sin t, \cos t)$

$$\int_C e^{\sqrt{x^2+y^2}} ds = \int_0^1 e^{\sqrt{2 \cdot 2t^2}} \sqrt{2+2}\, dt + \int_0^{\pi/4} e^2\, 2\sqrt{\sin^2 t + \cos^2 t}\, dt$$

$$= 2\int_0^1 e^{2t} dt + 2\int_{\pi/4}^{0} e^2 dt =$$

$$= 2 \left.\frac{e^{2t}}{2}\right|_0^1 + \frac{\pi}{2} e^2 = e^2 - 1 + \frac{\pi}{2} e^2$$

Problem 4. Find the work done by the force field $\overrightarrow{F} = (z, x, y)$ when moving a particle along the line segment connecting the points $(3, 0, 0)$ and $(0, \pi/2,\ 3)$.

$$W = \int_C \vec{F} \cdot \vec{dr}$$

$$\vec{r}(t) = (3,0,0)\cdot(1-t) + (0, \pi/2, 3)\cdot t =$$

$$= (3-3t\ ,\ \pi/2\, t\ ,\ 3t)$$

$$\vec{r}\,'(t) = (-3, \pi/2, 3) \qquad 0 \le t \le 1$$

$$W = \int_0^1 \underset{(3t,\ 3-3t,\ \pi/2\, t)}{(z, x, y)} \cdot (-3, \pi/2, 3)\, dt =$$

$$= \int_0^1 \left(-9t + \pi/2\,(3-3t) + \pi/2\; 3t\right) dt =$$

$$= \int_0^1 \left(-9t + 3\pi/2\right) dt =$$

$$= -\frac{9}{2} + \frac{3\pi}{2} = \frac{3\pi - 9}{2} = \frac{3}{2}(\pi - 3)$$

Problem 5. Use the Fundamental theorem of Calculus for line integrals to compute

$$\int_C \frac{ydx - xdy}{x^2},$$

where C is a curve connecting the points $(2,1)$ and $(1,2)$, and not intersecting the y-axis.

$$\vec{F} = \left(\frac{y}{x^2}, -\frac{1}{x}\right)$$

Let f be a potential function:

$$f_x = \frac{y}{x^2} \implies f = -\frac{y}{x} + h(y)$$

$$f_y = -\frac{1}{x} \implies f_y = -\frac{1}{x} + h'(y) = -\frac{1}{x}$$

$$\implies h(y) = \text{const}$$

$$\implies f = -\frac{y}{x}$$

By FTC,

$$\int_C \frac{y\,dx - x\,dy}{x^2} = f(1,2) - f(2,1) =$$

$$= -\frac{2}{1} - \left(-\frac{1}{2}\right) = -2 + \frac{1}{2} = -\frac{3}{2}$$