Problem 1. Find the volume of the solid bounded by the parabolic cylinder $y = x^2$ and the planes $z = 3y$, $z = 2 + y$.

$$-\int_{-1}^{1}\int_{x^2}^{1}\int_{2+y}^{3y} dz\,dy\,dx = \frac{16}{15}.$$

Problem 2. Use a change of variables to evaluate the integral

$$I = \int\int_R (4x^2 - y^2)\,dA,$$

where R is the region given by the inequality $|2x| + |y| \leq 2$.

$$u = y + 2x$$
$$v = y - 2x$$

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\frac{\partial(u,v)}{\partial(x,y)}\right|^{-1} = \frac{1}{\left|\begin{matrix} 2 & 1 \\ -2 & 1 \end{matrix}\right|} = \frac{1}{4}.$$

$$I = \int_{-2}^{2}\int_{-2}^{2} (-u \cdot v) \cdot \frac{1}{4}\,du\,dv = 0.$$

Problem 3. Find the volume of the solid above the surface $\varphi = \pi/3$ and below the surface $\rho = 4\cos\varphi$. (Here (ρ, θ, φ) are the standard spherical cooridnates).

$$\int_0^{\pi/3}\int_0^{2\pi}\int_0^{4\cos\varphi} \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi = \quad 10\pi.$$

Problem 4. Find the coordinates of the center of gravity of a semidisc $x^2 + y^2 \le R^2$, $y \ge 0$ if its density is proportional to the square of the distance to the x-axis, that is, $\rho(x, y) = ky^2$.

$\bar{x} = 0$ since the semidisc is symmetric w.r. to y-axis (and density distribution)

$$\text{mass} = \int_0^{\pi}\int_0^{R} k r^3 \sin^2\theta \, dr\, d\theta = k\frac{R^4\pi}{8}$$

$$M_x = \int_0^{\pi}\int_0^{R} k r^4 \sin^3\theta \, dr\, d\theta = \frac{kR^5}{5}\cdot\frac{4}{3}$$

$$\bar{y} = \frac{M_x}{\text{mass}} = \frac{32R}{15\pi}$$

Problem 5. Compute the surface integral $\int\int_S x^2z^2 dS$, where S is the part of the cone $z^2 = x^2 + y^2$ that lies between the planes $z = 1$ and $z = 3$.

$$\vec{r}(r,\theta) = (r\cos\theta,\ r\sin\theta,\ r),$$

$$D = \{(r,\theta) \mid 1 \le r \le 3,\ 0 \le \theta \le 2\pi\}.$$

$$|\bar{r}_\theta \times \bar{r}_r| = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = r\sqrt{2}$$

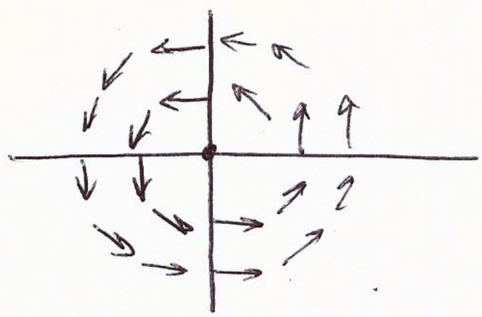
$$\int_0^{2\pi}\int_1^3 r^4\cos^2\theta \cdot \sqrt{2}\, r\, dr\, d\theta = \frac{364\sqrt{2}}{3}\pi.$$

Problem 6. Let $\mathbf{F} = \frac{(-y,x)}{x^2+y^2} = \frac{-y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j}$ be a vector field on the (x,y)-plane.

(1) Sketch $\mathbf{F}$.
(2) Is the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ path independent? (If it is, prove this. If not, give an example showing it).
(3) Is there a function f such that $\nabla f = \mathbf{F}$? Justify your answer.

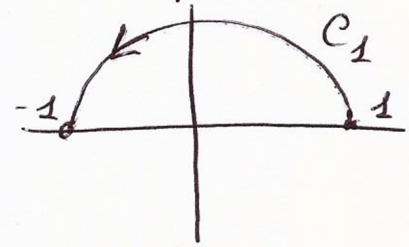
(1)

$\vec{F}$:

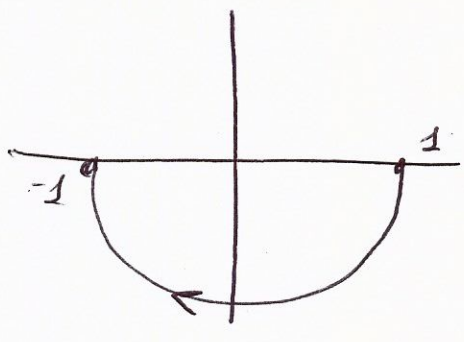


(2) The integral is not path independent.

Let C_1 be the curve (semicircle)



Let C_2 be



$$\int_{C_1} \vec{F} d\vec{r} \neq \int_{C_2} \vec{F} d\vec{r}$$

(3) $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ on $\mathbb{R}^2 \setminus \{0,0\}$ (which is not simply connected!)

However, since path-independence property is not satisfied (see part (2)), $\vec{F}$ is not conservative.

Problem 7. 1. Prove the formula for the area of the region D bounded by a simple closed positively-oriented curve C:

$$\text{Area of } D = \frac{1}{2}\int_C x\,dy - y\,dx.$$

2. Sketch the curve given by $x = t - \sin t$, $y = 1 - \cos t$. (This curve is called the cycloid).
3. Use the formula above to compute the area under one arch of the cycloid.

1. $\text{Area} = \iint_D 1 \cdot dA$

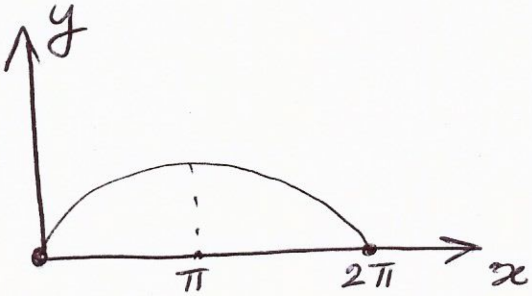
Green's thm: $\iint_D (Q_x - P_y)\,dA = \int_C P\,dx + Q\,dy$

If $Q_x - P_y = 1$,

take $P = x/2$, $Q = -y/2$.

Green's thm implies $\text{Area} = \frac{1}{2}\int_C x\,dy - y\,dx$

2.



3. $x = t - \sin t$ $\qquad y = 1 - \cos t$

$dy = \sin t\,dt$ $\qquad dx = dt - \cos t\,dt$

$$\text{Area} = \frac{1}{2}\int_0^{2\pi} (t - \sin t)\sin t\,dt - \frac{1}{2}\int_0^{2\pi} (1-\cos t)(1-\cos t)\,dt$$

Problem 8. Calculate the surface integral

$$\int\int_S \frac{\mathbf{r}}{|\mathbf{r}|} dS,$$

where S is the surface consisting of the hemishere $x^2 + y^2 + z^2 = 1$, $z \geq 0$ and the disk $x^2 + y^2 = 1$ on the xy-plane, oriented outward.

Let S_1 be the hemisphere and S_2 be the disk.

On S_1 : $\vec{n} = \frac{\vec{r}}{|\vec{r}|}$, so that $\frac{\vec{r}}{|\vec{r}|} \cdot \vec{n} = 1$.

$$\iint_{S_1} \frac{\vec{r}}{|\vec{r}|} d\vec{S} = \iint_{S_1} dS = \text{Area of hemisphere} = 2\pi.$$

On S_2: $\vec{n} = -\vec{k} \perp \vec{r}$, so that $\vec{n} \cdot \vec{r} = 0$

$$\Rightarrow \iint_{S_2} \frac{\vec{r}}{|\vec{r}|} d\vec{S} = 2\pi.$$

Answer: 2π.

Problem 9. Use Stokes' theorem to evaluate $\int\int_S \text{curl}\mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = x^2yz \cdot \mathbf{i} + yz^2 \cdot \mathbf{j} + z^3e^{xy} \cdot \mathbf{k}$, where S is the part of the sphere $x^2 + y^2 + z^2 = 5$ that lies above the plane $z = 1$, and S is oriented downward.

$I = \iint_S \text{curl}\vec{F}\, d\vec{S} = \int_C \vec{F}\, d\vec{r}$, where C is the circle of intersection of $x^2+y^2+z^2=5$ and $z=1$.

because of given orientation

$C: \quad \vec{r} = -(2\cos\theta,\ 2\sin\theta, 1)$

$\vec{r}\,' = -(-2\sin\theta,\ 2\cos\theta, 0)$

$$I = \int_0^{2\pi} (-2\sin 2\theta + 4\sin^2 2\theta)\, d\theta = 4\pi.$$

Problem 10. Let F be an inverse square field, that is, $\mathbf{F}(\mathbf{r}) = c\frac{\mathbf{r}}{|\mathbf{r}|^3}$ for some constant c. Show that the flux of $\mathbf{F}$ across a sphere S with center at the origin is independent of the radius of S.

Let R be the radius.

$$\text{Flux of } F = \iint_S \vec{F} \cdot \vec{n}\, dS$$

Since $\vec{n} = \frac{\vec{r}}{|\vec{r}|}$ on the sphere,

$$\vec{F} \cdot \vec{n} = c \cdot \frac{(\vec{r} \cdot \vec{r})}{|\vec{r}|^4} = \frac{c}{|\vec{r}|^2}$$

Parametrize S by θ, φ :

$|r_\theta \times r_\varphi| = |\vec{r}|^2$.

$\Rightarrow$ Under the integral you have $\frac{c}{|\vec{r}|^2} \cdot |\vec{r}|^2$ $\Rightarrow$

$$\text{Flux} = \underbrace{4\pi}_{\text{surface area}} \cdot c \quad \text{- independent of } R.$$