MATH 32B
FIRST MIDTERM EXAMINATION

October, 22nd, 2007

Please show your work. You will receive little or no credit for a correct answer to a problem which is not accompanied by sufficient explanations. If you have a question about any particular problem, please raise your hand and one of the proctors will come and talk to you. At the completion of the exam, please hand the exam booklet to your TA. If you have any questions about the grading of the exam, please see the instructor *within 15 calendar days of the examination.*

Name:________________________ Section:________________

#1	#2	#3	#4	#5		Total

Problem 1. Compute the integral $I = \int\int_R \frac{xy^2}{x^2+1} dA$, where $R = [0,1] \times [-3,3]$.

SOLUTION:

$$\begin{aligned} I = \int_{-3}^{3} y^2 \left(\int_0^1 \frac{xdx}{x^2+1} \right) dy &= \frac{1}{2} \int_{-3}^{3} y^2 \int_1^2 \frac{du}{u} \, dy = \\ = \frac{1}{2} \ln u|_1^2 \cdot \int_{-3}^{3} y^2 dy &= \frac{\ln 2}{2} \cdot \frac{y^3}{3}|_{-3}^{3} = \\ = \frac{\ln 2}{2} \cdot \frac{2 \cdot 3^3}{3} &= 9 \ln 2. \end{aligned}$$

Above $u = x^2 + 1$.

Problem 2. Find the volume of the solid bounded by the surfaces $y = x^2$, $y = 1$, $z = 0$ and $z = x^2 + y^2$.

Solution: The volume of the solid in questions is represented by the integral of the function $(x^2 + y^2)$ over the region $D = \{(x, y) | \ 0 \leq x \leq 1, \ x^2 \leq y \leq 1\}$:

$$\int_0^1 \int_{x^2}^1 (x^2 + y^2) dy \ dx = \int_0^1 x^2 \cdot (1 - x^2) + \frac{1}{3}(1 - x^6) dx =$$

$$\frac{x^3}{3}\Big|_0^1 - \frac{x^5}{5}\Big|_0^1 + \frac{1}{3} - \frac{1}{21}x^7\Big|_0^1 = \frac{1}{3} - \frac{1}{5} + \frac{1}{3} - \frac{1}{27} = \frac{17}{27} - \frac{1}{5} = \frac{58}{135}$$

Problem 3. Compute the integral

$$\int_0^{\frac{1}{\sqrt{2}}} \int_x^{\sqrt{1-x^2}} \arctan\left(\frac{y}{x}\right) dy\ dx$$

SOLUTION: Sketch the region of integration. Observe that it is 1/8th of a circle of radius 1, located between $\theta = \pi/4$ and $\theta = \pi/2$. Convert the integral into polar coordinates:

$$\begin{aligned}
I &= \int_{\pi/4}^{\pi/2} \int_0^1 \theta \cdot r dr d\theta = \\
&= \frac{\theta^2}{2}\Big|_{\pi/4}^{\pi/2} \cdot \frac{r^2}{2}\Big|_0^1 = \\
&= \frac{1}{4}\pi^2(1/4 - 1/16) = \\
&= \frac{1}{4} \cdot \frac{3}{16}\pi^2 = \frac{3}{64}\pi^2.
\end{aligned}$$

Problem 4. Find the volume of the part of the sphere $x^2 + y^2 + z^2 = 9z$ that lies inside of the paraboloid $z = x^2 + y^2$.

SOLUTION: Put the equation of the sphere into the standard form:

$$\begin{aligned} x^2 + y^2 + z^2 - 2 \cdot \frac{9}{2} z + \frac{81}{4} &= \frac{81}{4} \\ x^2 + y^2 + (z - \frac{9}{2})^2 &= \frac{9^2}{2^2} \end{aligned}$$

So, the sphere has radius $\frac{9}{2}$ and is centered at $(0, 0, \ 9/2)$. The two surfaces intersect at the points where $z + z^2 = 9z$, i.e., $z = 0$ and $z = 8$. When $z = 8$, $r = \sqrt{8} = 2\sqrt{2}$.

For the top part of the sphere we have

$$\begin{aligned} z - \frac{9}{2} &= \sqrt{\left(\frac{9}{2}\right)^2 - x^2 - y^2} \\ z &= \frac{9}{2} + \sqrt{\left(\frac{9}{2}\right)^2 - x^2 - y^2} \end{aligned}$$

Switching to polar coordinates, we get that the volume of the part of the sphere inside of the parabaloid is given by

$$\begin{aligned} \int_0^{2\pi} \int_0^{2\sqrt{2}} \left(\frac{9}{2} + \sqrt{\left(\frac{9}{2}\right)^2 - r^2} - r^2 \right) r dr d\theta &= 2\pi \int_0^{2\sqrt{2}} \left(\frac{9}{2} + \sqrt{\left(\frac{9}{2}\right)^2 - r^2} - r^2 \right) r dr \\ &= \pi \int_0^8 \left(\frac{9}{2} + \sqrt{\left(\frac{9}{2}\right)^2 - u} - u \right) du, \end{aligned}$$

where $u = r^2$. Finish the computation of the integral to get the final answer.

Problem 5. A student wants to have a sandwich and a cup of coffee for lunch. He is getting the sandwich at Slow Serve Sandwich Shop, where the average time to get a sandwich equals to $\mu = 1\ hr$. After that, the student goes to Confused Coffeemakers, where on average it takes $\mu = 1\ hr$ to get a cup of coffee. What is the probability that if a student goes for lunch at 11 a.m. he will get it before 1 p.m.? (Your answer should be a number rounded to the nearest 10%).

SOLUTION: Let X be the random variable representing the time it takes to get a sandwich and Y be the random variable representing the time it takes to get a cup of coffee. The distribution functions for these random variables are given by the same expression:

$$f(t) = \begin{cases} 0, & t \leq 0 \\ e^{-t}, & t > 0 \end{cases}$$

One can take μ to be equal to 1, since the average waiting time is given to be 1 hr.

The condition that the student who starts at 11 a.m. finishes before 1 hr. means that $X + Y \leq 2$. Thus, we are integrating over the triangle on the (x, y) plane whose coordinates are $(0, 0)$, $(2, 0)$ and $(0, 2)$. The probability is expressed by the integral=

$$\begin{aligned} P(X + Y \leq 2) = & \\ & = \int_0^2 \int_0^{2-x} e^{-y} \cdot e^{-x} dy dx = \\ & = -\int_0^2 e^{-x} \cdot (e^{-y})|_0^{2-x} dx = \\ & = -\int_0^2 e^{-x}(e^{x-2} - 1) dx = \\ & = -\int_0^2 (e^{-2} - e^{-x})\ dx = \\ & = \int_0^2 (e^{-x} - e^{-2}) dx = \\ & = -e^{-x}|_0^2 - e^{-2} \cdot x|_0^2 = \\ & = 1 - e^{-2} - 2e^{-2} = 1 - \frac{3}{e^2} \end{aligned}$$

Since $e \approx 2.78 \approx 3$, $\frac{3}{e^2} \approx \frac{1}{3}$, and the probability is approximately 0.66, or around 70% (the answer 60% also counts as correct if you have the same exact formula for the probability but use a different approximation for $\frac{3}{e^2}$).