Exam 1, Version A

Math 20D, Lecture C, Winter 2006
6 February 2006

Name: ______________________________

ID #: ______________________________

Section Time: _________________________

#	Score
1a	
1b	
1c	
2	
3a	
3b	
3c	
4a	
4b	
Total	

Read all of the following information before starting the exam:

- Show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Justify your answers algebraically whenever possible to ensure full credit. When you do use your calculator, sketch all relevant graphs and explain all relevant mathematics.
- Circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point. I will take points off for rambling and for incorrect or irrelevant statements.
- This test has 4 problems and is worth 100 points. It is your responsibility to make sure that you have all of the pages!
- Good luck!

1. [*24 points*] Determine whether each of the following equations is linear, separable, exact, or none of the above – in other words, identify a solution approach for the problem. Some of these may satisfy multiple labels – pick one. *Do not solve the differential equation.*

a. [*8 pts*]

$$\frac{dy}{dx} = xy^3(1+x^2)^{-1/2}$$

Solution:

We re-write the equation as

$$y^{-3}\frac{dy}{dx} - x(1+x^2)^{-1/2} = 0.$$

This is in separable form, so the equation is **separable**. *Moreover, we note that $M_y(x) = N_x(y) = 0$, so it is also* **exact**. *The equation is not linear.*

b. [*8 pts*]

$$\frac{dy}{dx} = \frac{3x^2 - 2xy + 2}{x^2 - 6y^2 - 3}$$

Solution:

We multiply through by the denominator of the right-hand side to get

$$(x^2 - 6y^2 - 3)\frac{dy}{dx} + (2xy - 3x^2 - 2) = 0$$

This is in exact form, but first we must check that it is exact. Differentiating the first term by x and the second by y we get

$$\frac{d}{dx}(x^2 - 6y^2 - 3) = 2x = \frac{d}{dy}(2xy - 3x^2 - 2).$$

Thus the equation is **exact**. *It is neither separable nor linear.*

c. [*8 pts*]

$$\frac{dy}{dx} = 2xe^{-x^2} - 2xy$$

Solution:

This equation is already in linear form, hence it is **linear**. *It is neither exact nor separable.*

2. [*25 points*] Compute the *explicit* solution to the following initial value problem:

$$\frac{dy}{dx} = \frac{2x}{1+2y}\,, \qquad y\left(\tfrac{1}{2}\right) = 0$$

Solution:

This is a separable equation,

$$(1+2y)\frac{dy}{dx} - 2x = 0.$$

Integrating both terms with respect to x, we get

$$\int (1+2y)\frac{dy}{dx}dx - \int 2x dx + c = 0,$$

which simply becomes

$$\int (1+2y)dy - \int 2x dx + c = 0.$$

Integrating, we get the general solution

$$y + y^2 - x^2 + c = 0.$$

Using the initial conditions, $y\left(\tfrac{1}{2}\right) = 0$, we may solve for c:

$$(0) + (0)^2 - \left(\frac{1}{2}\right)^2 + c = 0 \quad \Rightarrow \quad c = \frac{1}{4}.$$

Thus the implicit solution to our initial value problem is

$$y^2 + y - x^2 + \frac{1}{4} = 0.$$

For the explicit solution, we must solve for y using the quadratic formula:

$$y = \frac{-1 \pm \sqrt{1 - 4(1)(\frac{1}{4} - x^2)}}{2(1)} = -\frac{1}{2} \pm x.$$

Only one of these two solutions satisfies the initial condition, $y\left(\tfrac{1}{2}\right) = 0$,

$$y = -\frac{1}{2} + x.$$

3. [*26 points*] This problem concerns the second-order, constant-coefficient, homogeneous differential equation

$$y'' - 9y' + 14y = 0.$$

a. [*10 pts*] Compute two solutions to the differential equation. *Solution:*

For these problems, we assume a solution of the form $y(t) = e^{rt}$*. Inserting this into the differential equation we have the characteristic equation*

$$r^2 - 9r + 14 = 0$$

which is solved for $r = \{2, 7\}$*, thus our two solutions are*

$$y_1(t) = e^{2t}, \quad y_2(t) = e^{7t}.$$

We may check these individually to see that they indeed both solve the differential equation.

b. [*8 pts*] Show that these constitute a fundamental set of solutions (*Hint: consider the Wronskian,* $W(y_1, y_2) = y_1 y_2' - y_1' y_2$). *Solution:*

As discussed in the text, the solutions $y_1(t)$ *and* $y_2(t)$ *form a fundamental set of solutions if their Wronskian is nonzero. Checking this, we see that*

$$W(y_1, y_2) = y_1 y_2' - y_1' y_2 = e^{2t}(7e^{7t}) - 2e^{2t}(e^{7t}) = 7e^{9t} - 2e^{9t} = 5e^{9t} \neq 0.$$

c. [*8 pts*] Find the particular solution to this equation satisfying the initial conditions

$$y(0) = 2, \quad y'(0) = 0$$

Solution:

Since y_1 *and* y_2 *form a fundamental set of solutions, the general solution to our second-order homogeneous differential equation is*

$$y(t) = c_1 e^{2t} + c_2 e^{7t} \quad \Rightarrow \quad y'(t) = 2c_1 e^{2t} + 7c_2 e^{7t}.$$

Inserting our initial conditions, we have the following system of algebraic equations for our coefficients c_1 *and* c_2*:*

$$\begin{array}{ccccccc} 2 & = & y(0) & = & c_1 e^0 + c_2 e^0 & = & c_1 + c_2 \\ 0 & = & y'(0) & = & 2c_1 e^0 + 7c_2 e^0 & = & 2c_1 + 7c_2 \end{array}$$

which has solution $c_1 = \frac{14}{5}, c_2 = -\frac{4}{5}$*, so the particular solution satisfying our initial conditions is*

$$y(t) = \tfrac{14}{5} e^{2t} - \tfrac{4}{5} e^{7t}.$$

4. [*25 points*] A body falling in a dense fluid, such as oil, is acted on by three forces: a downward force due to gravity (weight), an upward buoyant force, and a resistive force acting opposite to the motion of the body. The size of the buoyant force, B, is equal to the gravitational weight of the fluid displaced by the object. For a slowly-moving spherical body of radius a, the resistive force is given by Stokes' Law, $R = 6\pi\mu a|v|$, where v is the velocity of the body and μ is the coefficient of viscosity for the surrounding fluid.

a. [*15 points*] Formulate the initial value problem for the velocity of a slowly-moving object having density ρ, where $\rho > \rho_f$, with ρ_f the density of the surrounding fluid. Assume that the object begins with zero initial velocity, and that the fluid has infinite depth.

b. [*10 points*] Determine the limiting velocity, v_l, of the body in terms of the densities ρ and ρ_f, the viscosity μ, the radius a, and the gravitational constant g.

Hints:
(i) the volume for a sphere of radius a is given by $V = \frac{4\pi a^3}{3}$.
(ii) mass is given by volume multiplied by density.
(iii) the gravitational weight is given as $f_g = m\,g$, for mass m and gravitational constant g.
(iv) Newton's Law of motion is written as **force = (mass)(acceleration)**.

Solution: The governing equation is $ma = G - B - R$, where G is the downward force due to gravity, B is the upward force due to buoyancy, and R is the upward force against the motion (since the initial velocity is zero and $\rho > \rho_f$ the motion will be downward, so R is upward). Defining terms, we have

- *The mass of the body is $m = V\rho = \frac{4}{3}\pi a^3 \rho$*
- *The acceleration is the time-derivative of the velocity, $a = \frac{dv}{dt}$*
- *The force due to gravity is $G = mg = \frac{4}{3}\pi a^3 \rho g$*
- *The resistive force is $R = 6\pi\mu a v$ (since the body is never rising, so $v > 0$)*
- *The buoyant force is $B = \frac{4}{3}\pi a^3 \rho_f g$, where we account for ρ_f in computation of the displaced fluid's mass (note, the body displaces a volume of fluid equal to its own volume).*

Combining these, we have the initial value problem

$$\left(\tfrac{4}{3}\pi a^3 \rho\right)\frac{dv}{dt} = \tfrac{4}{3}\pi a^3 g(\rho - \rho_f) - 6\pi\mu a v, \quad v(0) = 0,$$

or more simply expressed as

$$\frac{dv}{dt} = g\left(1 - \frac{\rho_f}{\rho}\right) - \left(\frac{9\mu}{2\rho a^2}\right)v, \quad v(0) = 0.$$

The limiting velocity, v_l may be found as the single equilibrium solution to the model, given as the solution to the equation

$$0 = g\left(1 - \frac{\rho_f}{\rho}\right) - \left(\frac{9\mu}{2\rho a^2}\right)v_l,$$

which gives the limiting velocity

$$v_l = \frac{2a^2 g}{9\mu}(\rho - \rho_f).$$