Chapter 3. Applications of Derivatives

3.3. The Shape of a Graph

Definition. Let f be a function defined on an interval I. Then

1. f *increases* on I if for all points x_1 and x_2 in I, $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$.

2. f *decreases* on I if for all points x_1 and x_2 in I, $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$.

Corollary 3. The First Derivative Test for Increasing and Decreasing.

Suppose that f is continuous on $[a, b]$ and differentiable on (a, b)

If $f' > 0$ at each point of (a, b), then f increases on $[a, b]$.

If $f' < 0$ at each point of (a, b), then f decreases on $[a, b]$.

Proof. Suppose $x_1, x_2 \in [a, b]$ with $x_1 < x_2$. The Mean Value Theorem applied to f on $[x_1, x_2]$ implies that $f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$ for some c between x_1 and x_2. Since $x_2 - x_1 > 0$, then $f(x_2) - f(x_1)$ and

$f'(c)$ are of the same sign. Therefore $f(x_2) > f(x_1)$ if f' is positive on (a, b), and $f(x_2) < f(x_1)$ if f' is negative on (a, b). *QED*

Example. Page 255 number 18 a,b.

Note. First Derivative Test for Local Extrema.

At a critical point $x = c$,

1. f has a *local minimum* if f' changes from negative to positive at c

2. f has a *local maximum* if f' changes from positive to negative at c

3. f has *no local extreme* if f' has the sign on both sides of c.

Example. Page 255 number 18 e.

Definition. The graph of a differentiable function $y = f(x)$ is

(a) *concave up* on an open interval I if y' is increasing on I

(b) *concave down* on an open interval I if y' is decreasing on I.

Note. Second Derivative Test for Concavity.

The graph of a twice-differentiable function $y = f(x)$ is

(a) concave up on any interval where $y'' > 0$

(b) concave down on any interval where $y'' < 0$.

Note. If f is concave up at point (x_0, y_0), then a tangent line to f at (x_0, y_0) lies **below** the graph of f near (x_0, y_0). If f is concave down at point (x_0, y_0), then a tangent line to f at (x_0, y_0) lies **above** the graph of f near (x_0, y_0).

Definition. A point where the graph of a function has a tangent line and where the concavity changes is a *point of inflection.*

Example. Page 255 number 34.

Theorem 5. Second Derivative Test for Local Extrema.

1. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.

2. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.

Example. Page 254 number 6, page 256 number 50.