# **Chapter 3.** Applications of Derivatives

## **3.2.** The Mean Value Theorem and Differential Equations

**Theorem 3. Rolle's Theorem.**

Suppose that $y = f(x)$ is continuous at every point of $[a, b]$ and differentiable at every point of $(a, b)$. If $f(a) = f(b) = 0$, then there is at least one number $c$ in $(a, b)$ at which $f'(c) = 0$.

**Proof.** Since $f$ is continuous by hypothesis, $f$ assumes an absolute maximum and minimum for $x \in [a, b]$ by Theorem 1 (the Extreme Value Theorem). These extrema occur only

**1.** at interior points where $f'$ is zero

**2.** at interior points where $f'$ does not exist

**3.** at the endpoints of the function's domain, $a$ and $b$.

Since we have hypothesized that $f$ is differentiable on $(a, b)$, then Option 2 is not possible.

In the event of Option 1, the point at which an extreme occurs, say $c$, must satisfy $f'(c) = 0$ by Theorem 2 of Section 3.1 (Local Extreme Values). Therefore the theorem holds.

In the event of Option 3, the maximum and minimum occur at the endpoints $a$ and $b$ (where $f$ is 0) and so $f$ must be a constant of 0 throughout the interval. Therefore $f'(x) = 0$ for all $x \in (a, b)$, by Rule 1 page 149, and the theorem holds. *QED*

**Example.** Page 244 number 38.

**Theorem 4. The Mean Value Theorem**

Suppose that $y = f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the interval $(a, b)$. Then there is at least one point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

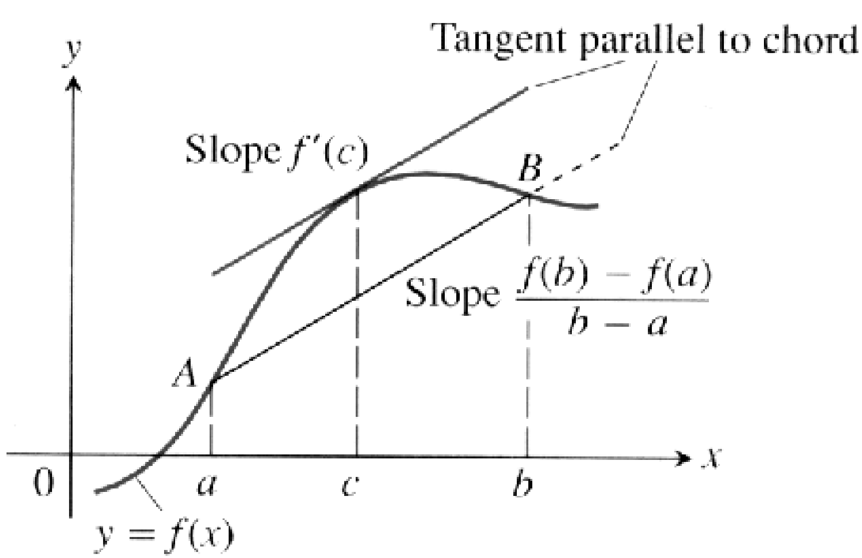


Figure 3.2.13, page 238

**Examples.** Page 244 numbers 32, 43.

**Corollary 1. Functions with Zero Derivatives Are Constant Functions.**

If $f'(x) = 0$ at each point of an interval $I$, then $f(x) = k$ for all $x \in I$, where $k$ is a constant.

**Note.** Corollary 1 is the *converse* of Rule 1 from page 149.

**Corollary 2. Functions with the Same Derivative Function on an Interval Differ by a Constant Value There**

If $f'(x) = g'(x)$ at each point of an interval $I$, then there exists a constant $k$ such that $f(x) = g(x) + k$ for all $x \in I$.

**Proof.** Consider the function $h(x) = f(x) - g(x)$. Under our hypothesis, $h(x)$ is constant on $I$ and so $h'(x) = 0$ for all $x \in I$. So by Corollary 1, $h(x) = k$ in $I$. Therefore $f(x) - g(x) = k$ and $f(x) = g(x) + k$. *QED*

**Example.** Page 243 number 14.

**Definition.** A *differential equation* is an equation relating an unknown function and one or more of its derivatives. A function whose derivatives satisfy a differential equation is called a *solution* of the differential equation.

**Example.** Page 244 number 22.