Linear Algebraic Equations

Linear versus nonlinear equation

A linear equation in two variables $x, \ y$

$$a\,x + b\,y = c$$

A nonlinear equation in two variables $x, \ y$

$$a\,x^2 + b\,y = c$$

Solution of linear equations is covered in this lecture. Solution of nonlinear equations will be considered in a following lecture.

System of linear equations

A linear equation in n variables $x_1, \ x_2, \ \ldots, \ x_n$

$$a_1\,x_1 + a_2\,x_2 + \ldots + a_n\,x_n = b$$

A system of m linear equations in n unknowns

$$A_{11}\,x_1 + A_{12}\,x_2 + \ldots + A_{1\,n}\,x_n = b_1$$

$$A_{21}\,x_1 + A_{22}\,x_2 + \ldots + A_{2\,n}\,x_n = b_2$$

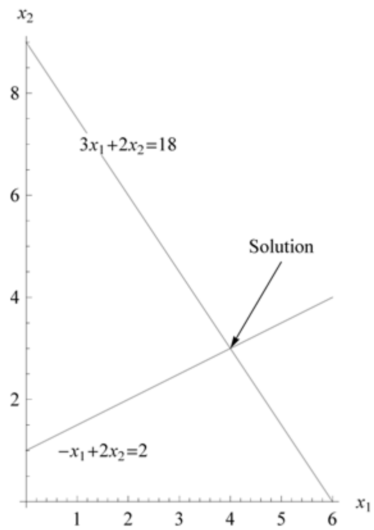
$$\vdots$$

$$A_{m\,1}\,x_1 + A_{m\,2}\,x_2 + \ldots + A_{m\,n}\,x_n = b_m$$

In matrix form: $$\begin{pmatrix} A_{11} & A_{12} & \ldots & A_{1\,n} \\ A_{21} & A_{22} & \ldots & A_{2\,n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m\,1} & A_{m\,2} & \ldots & A_{m\,n} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

Solution of linear equations

For a unique solution we must have as many equations as the number of unknowns.



Solution of a system of *n* linear equations in *n* unknowns written in matrix form

$$A_{11}\, x_1 + A_{12}\, x_2 + \ldots + A_{1\,n}\, x_n = b_1$$

$$A_{21}\, x_1 + A_{22}\, x_2 + \ldots + A_{2\,n}\, x_n = b_2$$

$$\vdots$$

$$A_{n\,1}\, x_1 + A_{n\,2}\, x_2 + \ldots + A_{n\,n}\, x_n = b_n$$

$$\begin{pmatrix} A_{11} & A_{12} & \ldots & A_{1\,n} \\ A_{21} & A_{22} & \ldots & A_{2\,n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n\,1} & A_{n\,2} & \ldots & A_{n\,n} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$$\Longrightarrow \quad \boldsymbol{A}\, \boldsymbol{x} = \boldsymbol{b}$$

Solution: $\boldsymbol{x} = \boldsymbol{A}^{-1}\, \boldsymbol{b}$

Using matlab

x = inv(A) * b or x = A\b

The backslash form is more efficient and should be used in general.

Class Activity 1

Determine solution of the following system of linear equations

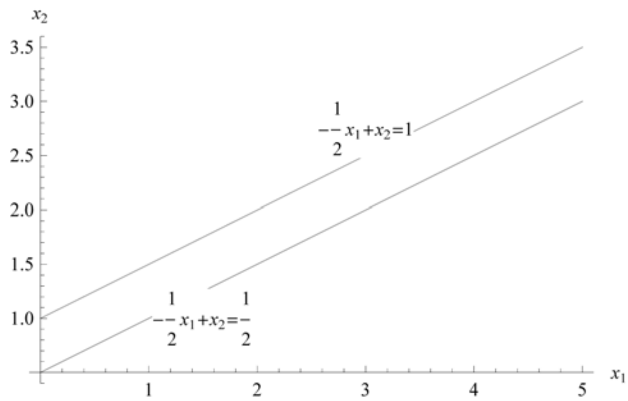
$8\,x_1 - 6\,x_2 + 2\,x_3 - 28 = 0$

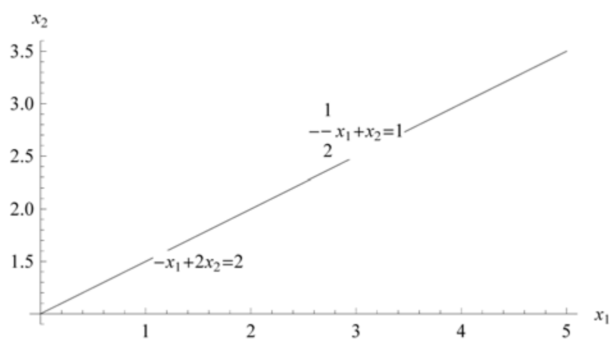
$-4\,x_1 + 11\,x_2 = 7\,x_3 - 40$

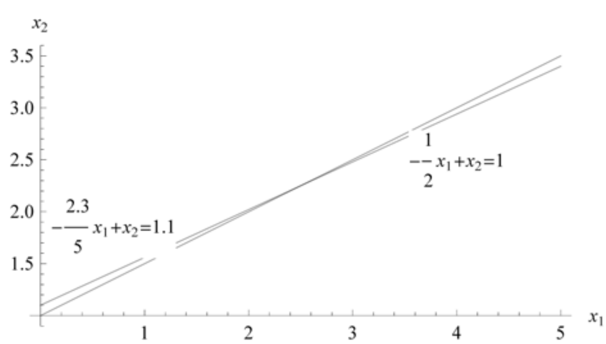
$4\,x_1 - 7\,x_2 + 6\,x_3 = 33$

Ill-conditioning

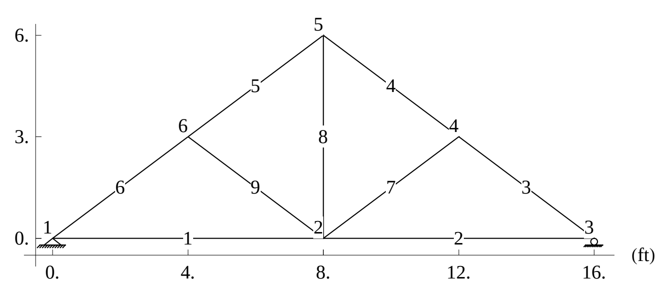
Even when we have the appropriate number of equations we may have no solution (a), infinitely many solutions (b), or ill-conditioning (c). In these situations the inverse of coefficient matrix $\boldsymbol{A}$ does not exist.





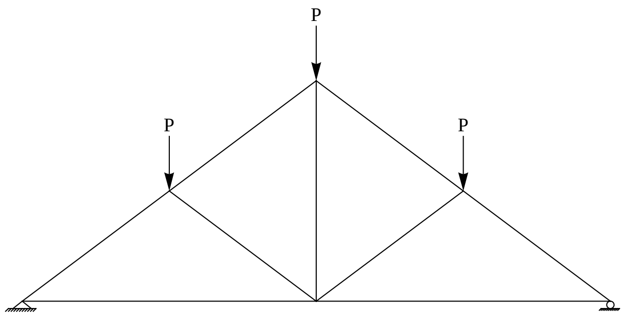


Application: Analysis of trusses using the finite element method



All members are made of $3.5\ \text{in} \times 3.5\ \text{in}$ construction grade lumber. The modulus of elasticity is $E = 1500\ \text{kip}/\text{in}^2$.

Loading due to roofing materials



The load $P = 1200$ lb. Using the given section dimensions and material modulus values in the finite element equations for truss elements the following system of equations is obtained relating the horizontal (u) and vertical (v) displacements at joints to the externally applied loading.

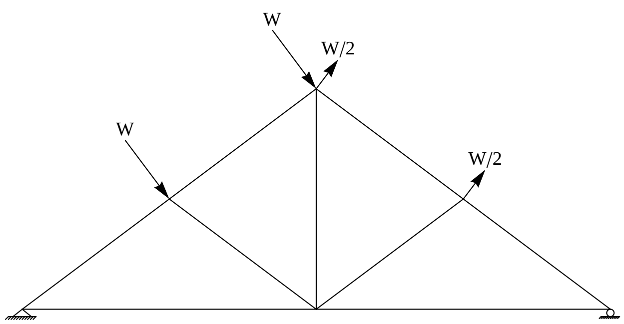
$$\begin{pmatrix} 774.813 & 0 & -191.406 & -196. & -147. & 0 & 0 & -196. & 147. \\ 0 & 475.708 & 0 & -147. & -110.25 & 0 & -255.208 & 147. & -110.25 \\ -191.406 & 0 & 387.406 & -196. & 147. & 0 & 0 & 0 & 0 \\ -196. & -147. & -196. & 588. & -147. & -196. & 147. & 0 & 0 \\ -147. & -110.25 & 147. & -147. & 330.75 & 147. & -110.25 & 0 & 0 \\ 0 & 0 & 0 & -196. & 147. & 392. & 0 & -196. & -147. \\ 0 & -255.208 & 0 & 147. & -110.25 & 0 & 475.708 & -147. & -110.25 \\ -196. & 147. & 0 & 0 & 0 & -196. & -147. & 588. & 147. \\ 147. & -110.25 & 0 & 0 & 0 & -147. & -110.25 & 147. & 330.75 \end{pmatrix} \begin{pmatrix} u_2 \\ v_2 \\ u_3 \\ u_4 \\ v_4 \\ u_5 \\ v_5 \\ u_6 \\ v_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1.2 \\ 0 \\ -1.2 \\ 0 \\ -1.2 \end{pmatrix}$$

```
A = [774.813, 0, -191.406, -196., -147., 0, 0, -196., 147.;
   0, 475.708, 0, -147., -110.25, 0, -255.208, 147., -110.25;
   -191.406, 0, 387.406, -196., 147., 0, 0, 0, 0;
   -196., -147., -196., 588., -147., -196., 147., 0, 0;
   -147., -110.25, 147., -147., 330.75, 147., -110.25, 0, 0;
   0, 0, 0, -196., 147., 392., 0, -196., -147.;
   0, -255.208, 0, 147., -110.25, 0, 475.708, -147., -110.25;
   -196., 147., 0, 0, 0, -196., -147., 588., 147.;
   147., -110.25, 0, 0, 0, -147., -110.25, 147., 330.75];
b = [0, 0, 0, 0, -1.2, 0, -1.2, 0, -1.2]';
A \ b
```

```
ans =

    0.0125
   -0.0486
    0.0251
    0.0047
   -0.0436
    0.0125
   -0.0439
    0.0204
   -0.0436
```

Loading due to wind



The load $W = 1800$ lb. Using the given section dimensions and material modulus values in the finite element equations for truss elements the following system of equations is obtained relating the horizontal (u) and vertical (v) displacements at joints to the externally applied loading.

$$\begin{pmatrix} 774.813 & 0 & -191.406 & -196. & -147. & 0 & 0 & -196. & 147. \\ 0 & 475.708 & 0 & -147. & -110.25 & 0 & -255.208 & 147. & -110.25 \\ -191.406 & 0 & 387.406 & -196. & 147. & 0 & 0 & 0 & 0 \\ -196. & -147. & -196. & 588. & -147. & -196. & 147. & 0 & 0 \\ -147. & -110.25 & 147. & -147. & 330.75 & 147. & -110.25 & 0 & 0 \\ 0 & 0 & 0 & -196. & 147. & 392. & 0 & -196. & -147. \\ 0 & -255.208 & 0 & 147. & -110.25 & 0 & 475.708 & -147. & -110.25 \\ -196. & 147. & 0 & 0 & 0 & -196. & -147. & 588. & 147. \\ 147. & -110.25 & 0 & 0 & 0 & -147. & -110.25 & 147. & 330.75 \end{pmatrix} \begin{pmatrix} u_2 \\ v_2 \\ u_3 \\ u_4 \\ v_4 \\ u_5 \\ v_5 \\ u_6 \\ v_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0.54 \\ 0.72 \\ -0.54 \\ -0.72 \\ -1.08 \\ -1.44 \end{pmatrix}$$

```
A = [774.813, 0, -191.406, -196., -147., 0, 0, -196., 147.;
   0, 475.708, 0, -147., -110.25, 0, -255.208, 147., -110.25;
   -191.406, 0, 387.406, -196., 147., 0, 0, 0, 0;
   -196., -147., -196., 588., -147., -196., 147., 0, 0;
   -147., -110.25, 147., -147., 330.75, 147., -110.25, 0, 0;
   0, 0, 0, -196., 147., 392., 0, -196., -147.;
   0, -255.208, 0, 147., -110.25, 0, 475.708, -147., -110.25;
   -196., 147., 0, 0, 0, -196., -147., 588., 147.;
   147., -110.25, 0, 0, 0, -147., -110.25, 147., 330.75];
b = [0, 0, 0, 0.54, 0.72, -0.54, -0.72, -1.08, -1.44]';
A \ b

ans =
    0.0053
   -0.0106
    0.0044
    0.0023
   -0.0016
   -0.0049
   -0.0116
    0.0024
   -0.0173
```

L U Decomposition

Provides an efficient way to solve equations with several different right-hand side vectors by separating the time consuming elimination of the matrix ***A*** from manipulations of the right-hand side ***b***. Also provides efficient means to compute the matrix inverse.

The key idea is to decompose the matrix ***A*** into two matrices ***L*** and ***U*** such that $\boldsymbol{L}\,\boldsymbol{U} = \boldsymbol{A}$ where ***L*** is a lower triangular matrix and ***U*** is an upper triangular matrix.

$$L = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ \ell_{2,1} & 1 & 0 & & 0 \\ \ell_{3,1} & \ell_{3,2} & 1 & & 0 \\ \vdots & \vdots & & \ddots & 0 \\ \ell_{n,1} & \ell_{n,2} & \ldots & \ell_{n,n-1} & 1 \end{pmatrix}$$

$$U = \begin{pmatrix} u_{1,1} & u_{1,2} & u_{1,3} & \ldots & u_{1,n} \\ 0 & u_{2,2} & u_{2,3} & \ldots & u_{2,n} \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & & & u_{n-1,n} \\ 0 & 0 & \ldots & & u_{n,n} \end{pmatrix}$$

Matlab function: [L,U] = **lu**(A) returns an upper triangular matrix in U and a permuted lower triangular matrix in L such that A = L*U.

Example

$3\,x_1 - 0.1\,x_2 - 0.2\,x_3 = 7.85$

$0.1\,x_1 + 7\,x_2 - 0.3\,x_3 = -19.3$

$0.3\,x_1 - 0.2\,x_2 + 10\,x_3 = 71.4$

```
A =[3.0000   -0.1000   -0.2000;
    0.1000    7.0000   -0.3000;
    0.3000   -0.2000   10.0000];
[L, U] = lu(A)

L =
    1.0000         0         0
    0.0333    1.0000         0
    0.1000   -0.0271    1.0000
U =
    3.0000   -0.1000   -0.2000
         0    7.0033   -0.2933
         0         0   10.0120

Verify factorization

L*U

ans =
    3.0000   -0.1000   -0.2000
    0.1000    7.0000   -0.3000
    0.3000   -0.2000   10.0000
```

Use of LU form in finding solution

$\boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{b}$
$\boldsymbol{L}\,\boldsymbol{U}\,\boldsymbol{x} = \boldsymbol{b}$
Two systems of linear equations

$\boldsymbol{U}\,\boldsymbol{x} = \boldsymbol{d}$

$\boldsymbol{L}\,\boldsymbol{d} = \boldsymbol{b}$

$\boldsymbol{L}\,\boldsymbol{d} = \boldsymbol{b}$ is used to generate an intermediate vector d by forward substitution. Then, $\boldsymbol{U}\,\boldsymbol{x} = \boldsymbol{d}$ is used to get x by back substitution.

```
L =
    1.0000         0         0
    0.0333    1.0000         0
    0.1000   -0.0271    1.0000
b = [7.8500;  -19.3000;   71.4000];
```

$\boldsymbol{L}\,\boldsymbol{d} = \boldsymbol{b}$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0.0333 & 1 & 0 \\ 0.1 & -0.0271 & 1 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} = \begin{pmatrix} 7.85 \\ -19.3 \\ 71.4 \end{pmatrix}$$

Forward substitution gives solution of this system of equations

$d_1 = 7.85$

$0.0333\,d_1 + d_2 = -\,19.3 \implies d_2 = -19.3 - 0.0333 \times 7.85 = -19.5617$

$0.1\,d_1 - 0.0271\,d_2 + d_3 = 71.4 \implies d_3 = 70.0843$

```
U =
    3.0000   -0.1000   -0.2000
         0    7.0033   -0.2933
         0         0   10.0120
```

$\boldsymbol{U}\,\boldsymbol{x} = \boldsymbol{d}$

$$\begin{pmatrix} 3 & -0.1 & -0.2 \\ 0 & 7.0033 & -0.2933 \\ 0 & 0 & 10.012 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7.85 \\ -19.5617 \\ 70.0843 \end{pmatrix}$$

Backward substitution gives solution of this system of equations

$$10.012\,x_3 = 70.0843 \implies x_3 = 7.0$$

$$7.0033\,x_2 - 0.2933\,x_3 = -19.5617 \implies x_2 = (-19.5617 + 0.2933 \times 7)/7.0033 = -2.5$$

$$3\,x_1 - 0.1\,x_2 - 0.2\,x_3 = 7.85 \implies x_3 = 3.0$$

Class Activity 2

Determine solution of the following two systems of linear equations

1: $x_1 - x_4 = 0$
$-x_1 + 2\,x_2 - x_3 = 0$
$-x_2 + 2\,x_3 - x_4 = 0$
$x_4 = 1$

2: $x_1 - x_4 = 1$
$-x_1 + 2\,x_2 - x_3 = 0$
$-x_2 + 2\,x_3 - x_4 = 0$
$x_4 = 0$

The only difference between the two systems is in the right-hand sides. Thus create a single LU factorization of the coefficient matrix and use only the forward and substitutions to obtain the final solutions.

Procedure for factorizing A into LU

To start with set matrix ***Z*** to be same as the matrix ***A***.

```
Z =
    3.0000   -0.1000   -0.2000
    0.1000    7.0000   -0.3000
    0.3000   -0.2000   10.0000

L11 = 1
L21 = Z21/Z11 = 0.1/3 = 0.0333
L31 = Z31/Z11 = 0.3/3 = 0.1000
```

To get the *Z* matrix for the next step, multiply row 1 by -L21 and add it to row 2. Multiply row 1 by -L31 and add it to row 3.

```
Z =
    3.0000   -0.1000   -0.2000
         0    7.0033   -0.2933
         0   -0.1900   10.0200

L22 = 1
L32 = Z32/Z22 = -0.19/7.0033 = -0.0271
```

To get the *Z* matrix for the next step, multiply row 2 by -L32 and add it to row 3.

```
Z =
    3.0000   -0.1000   -0.2000
         0    7.0033   -0.2933
         0         0   10.0120


L33 = 1
```

Since the given matrix *A* is a 3×3 matrix this is the end of the process. The last *Z* matrix is the upper triangular matrix ***U***.

```
U =
    3.0000   -0.1000   -0.2000
         0    7.0033   -0.2933
         0         0   10.0120
```

The factors used in various steps identify columns of the lower triangular matrix ***L***

```
L =
    1.0000         0         0
    0.0333    1.0000         0
    0.1000   -0.0271    1.0000
```

On p. 44 your textbook has matlab code (LUdec) that illustrates how you can implement this idea into a matlab function. For general use matlab's built-in function lu should obviously be used.

Need for pivoting

To get *Z* matrices for various steps we must divide by the diagonal element. Obviously if the diagonal element is 0 we cannot proceed. If the original system of equationshas a solution we can always find a nonzero diagonal term by re-arranging the system of equations. This is known as pivoting and is illustrated by the following example.

$$A = \begin{pmatrix} 2 & 4 & -2 & -2 \\ 1 & 2 & 4 & -3 \\ -3 & -3 & 8 & -2 \\ -1 & 1 & 6 & -3 \end{pmatrix}; \quad b = \begin{pmatrix} -4 \\ 5 \\ 7 \\ 7 \end{pmatrix}$$

To start with set matrix ***Z*** to be same as the matrix ***A***.

```
Z =
     2     4    -2    -2
     1     2     4    -3
    -3    -3     8    -2
    -1     1     6    -3
L11= 1
L21 = Z(2,1)/Z(1,1) = 0.5000
L31 = Z(3,1)/Z(1,1) = -1.5000
L41= Z(4,1)/Z(1,1) = -0.5000
```

To get the *Z* matrix for the next step, multiply row 1 by -L21 and add it to row 2. Multiply row 1 by -L31 and add it to row 3. Multiply row 1 by -L41 and add it to row 4.

```
Z =
     2     4    -2    -2
     0     0     5    -2
     0     3     5    -5
     0     3     5    -4

L22 = 1
L32 = Z32/Z22 Cannot compute since Z(2,2)= 0.
```

Since each row simply represents an equation a row interchange simply means re-ordering of equations. If we swap equations 2 and 3 we have

```
Z =
     2     4    -2    -2
     0     3     5    -5
     0     0     5    -2
     0     3     5    -4
```

Because of the row swap L21 and L31 are interchanges as well.

```
L31 = 0.5000
L21 = -1.5000

L22 = 1
L32 = Z32/Z22 = 0
L42 = Z42/Z22 = 1
```

To get the *Z* matrix for the next step, multiply row 2 by -L32 and add it to row 3. Multiply row 2 by -L42 and add it to row 4.

```
Z =
     2     4    -2    -2
     0     3     5    -5
     0     0     5    -2
     0     0     0     1

L33 = 1
L43 = Z43/Z33 = 0
```

To get the *Z* matrix for the next step, multiply row 3 by -L43 and add it to row 4.

```
Z =
     2     4    -2    -2
     0     3     5    -5
     0     0     5    -2
     0     0     0     1
```

```
L44 = 1
```

Since the given matrix *A* is a 4×4 matrix this is the end of the process. The last *Z* matrix is the upper triangular matrix ***U***.

```
U =
     2     4    -2    -2
     0     3     5    -5
     0     0     5    -2
     0     0     0     1
```

The factors used in various steps identify columns of the lower triangular matrix ***L***

```
L =
    1.0000         0         0         0
   -1.5000    1.0000         0         0
    0.5000         0    1.0000         0
   -0.5000    1.0000         0    1.0000
```

To compute the solution we must remember that the original order of the system of equations is changed in these factors. Thus the right hand side vector consistent with this factorization is

```
b = [-4; 7; 5; 7]
```

$\boldsymbol{L}\,\boldsymbol{d} = \boldsymbol{b} \implies \boldsymbol{d} = [-4;\ 1;\ 7;\ 4]$

$\boldsymbol{U}\,\boldsymbol{x} = \boldsymbol{d} \implies \boldsymbol{x} = [1;\ 2;\ 3;\ 4]$

On p. 71 your textbook has matlab code (LUdecPiv) that illustrates how you can implement this idea into a matlab function. For general use matlab's built-in function lu should obviously be used.

Using matlab function lu

The matlab function lu does perform the pivoting. However it re-arranges L and U matrices such that the right-hand side vectors still maintain the original order.

```
A =
     2     4    -2    -2
     1     2     4    -3
```

```
    -3    -3     8    -2
    -1     1     6    -3
b =
    -4
     5
     7
     7
[L, U] = lu(A)

L =
   -0.6667    1.0000         0         0
   -0.3333    0.5000    1.0000         0
    1.0000         0         0         0
    0.3333    1.0000    0.0000    1.0000
U =
   -3.0000   -3.0000    8.0000   -2.0000
         0    2.0000    3.3333   -3.3333
         0         0    5.0000   -2.0000
         0         0         0    1.0000

d = L\b

d =
    7.0000
    0.6667
    7.0000
    4.0000

x = U\d

x =
    1.0000
    2.0000
    3.0000
    4.0000
```

Application: Truss example using LU decomposition

```
A = [774.813, 0, -191.406, -196., -147., 0, 0, -196., 147.;
   0, 475.708, 0, -147., -110.25, 0, -255.208, 147., -110.25;
   -191.406, 0, 387.406, -196., 147., 0, 0, 0, 0;
   -196., -147., -196., 588., -147., -196., 147., 0, 0;
   -147., -110.25, 147., -147., 330.75, 147., -110.25, 0, 0;
   0, 0, 0, -196., 147., 392., 0, -196., -147.;
   0, -255.208, 0, 147., -110.25, 0, 475.708, -147., -110.25;
   -196., 147., 0, 0, 0, -196., -147., 588., 147.;
   147., -110.25, 0, 0, 0, -147., -110.25, 147., 330.75];
[L, U] = lu(A)
```

```
L =
  Columns 1 through 5
    1.0000         0         0         0         0
         0    1.0000         0         0         0
   -0.2470         0    1.0000         0         0
   -0.2530   -0.3090   -0.7186    1.0000         0
   -0.1897   -0.2318    0.3254   -0.4371    1.0000
         0         0         0   -0.6176    0.3395
         0   -0.5365         0    0.2147   -0.7728
   -0.2530    0.3090   -0.1424   -0.1227   -0.0243
    0.1897   -0.2318    0.1068    0.0921    0.0182
  Columns 6 through 9
         0         0         0         0
         0         0         0         0
         0         0         0         0
         0         0         0         0
         0         0         0         0
    1.0000         0         0         0
    0.3577    1.0000         0         0
   -0.8738    0.0815    1.0000         0
   -0.5200   -0.6871    0.4282    1.0000
U =
  Columns 1 through 5
  774.8130         0 -191.4060 -196.0000 -147.0000
         0  475.7080         0 -147.0000 -110.2500
         0         0  340.1220 -244.4189  110.6858
         0         0         0  317.3495 -138.7132
         0         0         0         0  180.6572
         0         0         0         0         0
         0         0         0         0         0
         0         0         0         0         0
         0         0         0         0         0
  Columns 6 through 9
         0         0 -196.0000  147.0000
         0 -255.2080  147.0000 -110.2500
         0         0  -48.4189   36.3142
 -196.0000   68.1374  -38.9509   29.2132
   61.3285 -139.6142   -4.3855    3.2891
  250.1279   89.4782 -218.5679 -130.0741
         0  184.2597   15.0247 -126.5960
         0         0  288.9989  123.7503
         0         0         0   63.0722

b = [0, 0, 0, 0, -1.2, 0, -1.2, 0, -1.2]';
d = L\b;
U\d
```

```
ans =

    0.0125
   -0.0486
    0.0251
    0.0047
   -0.0436
    0.0125
   -0.0439
    0.0204
   -0.0436

b = [0, 0, 0, 0.54, 0.72, -0.54, -0.72, -1.08, -1.44]';
d = L\b;
U\d

ans =
    0.0053
   -0.0106
    0.0044
    0.0023
   -0.0016
   -0.0049
   -0.0116
    0.0024
   -0.0173
```

Gauss-Seidel Method

Iterative or approximate methods provide an alternative to the elimination methods. The system of equations is reshaped by solving the first equation for x_1, the second equation for x_2, and the third for x_3, …. We can then start a numerical process by choosing an initial guess for the solution and refining it by using the equations.

$$\begin{pmatrix} 7 & 1 & 1 \\ -3 & 7 & -1 \\ -2 & 5 & 9 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ -26 \\ 1 \end{pmatrix}$$

Solving ith equation for the ith variable

$$x_1 = \frac{6 - x_2 - x_3}{7}$$

$$x_2 = \frac{-26 + 3\,x_1 + x_3}{7}$$

$$x_3 = \frac{1 + 2\,x_1 - 5\,x_2}{9}$$

Starting with $x_1 = x_2 = x_3 = 1$ the successive iterations yield

Iteration 1

$$x_1 = \frac{6 - x_2 - x_3}{7} = \frac{6 - 1 - 1}{7} = 0.571429$$

$$x_2 = \frac{-26 + 3\,x_1 + x_3}{7} = \frac{-26 + 3 \times 0.571429 + 1}{7} = -3.32653$$

$$x_3 = \frac{1+2\,x_1-5\,x_2}{9} = \frac{1+2\times0.571429-5\times-3.32653}{9} = 2.08617$$

Iteration 2

$$x_1 = \frac{6-x_2-x_3}{7} = \frac{6-(-3.32653)-(2.08617)}{7} = 1.03434$$

$$x_2 = \frac{-26+3\,x_1+x_3}{7} = \frac{-26+3\times1.03434+2.08617}{7} = -2.97297$$

$$x_3 = \frac{1+2\,x_1-5\,x_2}{9} = \frac{1+2\times1.03434-5\times-2.97297}{9} = 1.99262$$

Iteration 3

$$x_1 = \frac{6-x_2-x_3}{7} = \frac{6-(-2.97297)-(1.99262)}{7} = 0.997194$$

$$x_2 = \frac{-26+3\,x_1+x_3}{7} = \frac{-26+3\times0.997194+1.99262}{7} = -3.00226$$

$$x_3 = \frac{1+2\,x_1-5\,x_2}{9} = \frac{1+2\times0.997194-5\times-3.00226}{9} = 2.00063$$

Iteration 4

$$x_1 = \frac{6-x_2-x_3}{7} = \frac{6-(-3.00226)-(2.00063)}{7} = 1.00023$$

$$x_2 = \frac{-26+3\,x_1+x_3}{7} = \frac{-26+3\times1.00023+2.00063}{7} = -2.99981$$

$$x_3 = \frac{1+2\,x_1-5\,x_2}{9} = \frac{1+2\times1.00023-5\times-2.99981}{9} = 1.99995$$

Final solution: $x_1 = 1,\ \ x_2 = -3,\ \ x_3 = 2.$

Class Activity 3

Create a function gaussSeidel that takes coefficient matrix A, right-hand side vector b, and an initial guess vector x0 and returns a solution of the system using the Gauss-Seidel iteration. Test your function by finding the solution of the example used to illustrate the method.

Extra