# Chapter 2. Derivatives

## 2.4. Derivative of Trigonometric Functions

**Recall.** For all real numbers $a$ and $b$,

$$\sin(a+b) = \sin a \cos b + \cos a \sin b.$$

**Theorem. Derivative of the Sine Function**

$$\frac{d}{dx}[\sin x] = \cos x$$

**Proof.** Let $y = \sin x$. By definition we have

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h\to 0} \frac{(\sin x \cos h + \cos x \sin h) - \sin x}{h} \\
&= \lim_{h\to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h} \\
&= \lim_{h\to 0} \left(\sin x \cdot \frac{\cos h - 1}{h}\right) + \lim_{h\to 0} \left(\cos x \cdot \frac{\sin h}{h}\right) \\
&= \sin x \cdot \lim_{h\to 0} \frac{\cos h - 1}{h} + \cos x \cdot \lim_{h\to 0} \frac{\sin h}{h} \\
&= \sin x \cdot 0 + \cos x \cdot 1 \\
&= \cos x.
\end{aligned}$$

We have $\lim_{h\to 0} \frac{\cos h - 1}{h} = 0$ and $\lim_{h\to 0} \frac{\sin h}{h} = 1$ by the results in section 1.2.

*QED*

**Example.** Page 184 number 2.

**Recall.** For all real numbers $a$ and $b$ we have

$$\cos(a+b) = \cos a \cos b - \sin a \sin b.$$

**Theorem. Derivative of the Cosine Function**

$$\frac{d}{dx}[\cos x] = -\sin x$$

**Proof.** By definition we have

$$\begin{aligned}
\frac{d}{dx}[\cos x] &= \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h} \\
&= \lim_{h\to 0} \frac{(\cos x \cos h - \sin x \sin h) - \cos x}{h} \\
&= \lim_{h\to 0} \frac{\cos x(\cos h - 1) - \sin x \sin h}{h} \\
&= \lim_{h\to 0} \cos x \cdot \frac{\cos h - 1}{h} - \lim_{h\to 0} \sin x \cdot \frac{\sin h}{h} \\
&= \cos x \cdot \lim_{h\to 0} \frac{\cos h - 1}{h} - \sin x \cdot \lim_{h\to 0} \frac{\sin h}{h}
\end{aligned}$$

$$= \cos x \cdot 0 - \sin x \cdot 1$$

$$= -\sin x.$$

*QED*

**Examples.** Page 184 number 22, page 183 Example 5, and page 185 number 44.

**Note.** In summary, we have the following derivatives of the six trigonometric functions:

| $f$ | $f'$ |
|---|---|
| $\sin x$ | $\cos x$ |
| $\cos x$ | $-\sin x$ |
| $\tan x$ | $\sec^2 x$ |
| $\cot x$ | $-csc^2x$ |
| $\sec x$ | $\sec x \tan x$ |
| $\csc x$ | $-\csc x \cot x$ |

**Example.** Page 185 number 32.