Chapter 1. Limits and Continuity

1.4. Continuity

Definition. Continuity at a Point.

Interior Point: A function $y = f(x)$ is *continuous at an interior point* c of its domain if

$$\lim_{x \to c} f(x) = f(c).$$

Endpoint: A function $y = f(x)$ is *continuous at a left endpoint* a or is *continuous at a right endpoint* b of its domain if

$$\lim_{x \to a^+} f(x) = f(a) \quad \text{or} \quad \lim_{x \to b^-} f(x) = f(b), \quad \text{respectively.}$$

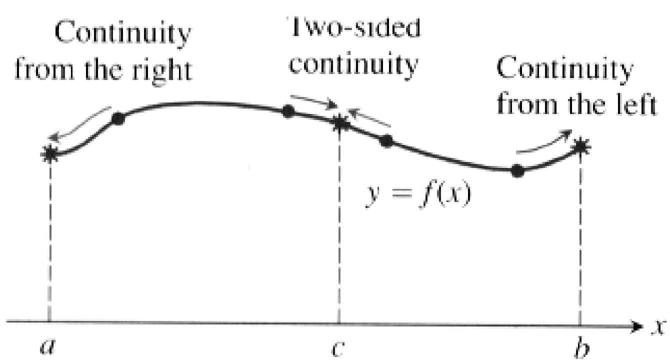

Figure 1.4.46, page 125.

Note. If a function is continuous at all interior points of its domain and the domain is an interval, then the function can be "drawn without picking up your pencil."

Example. Page 132 number 4.

Continuity Test.

A function $f(x)$ is continuous at an interior point of the domain of f, $x = c$ if and only if it meets the following three conditions:

1. $f(c)$ exists,

2. $\lim_{x \to c} f(x)$ exists, and

3. $\lim_{x \to c} f(x) = f(c)$.

Note. Polynomials, rational functions, and the six trigonometric functions are continuous at every point of their domains.

Example. Consider the piecewise defined function

$$f(x) = \begin{cases} x & \text{if } x \in (-\infty, 0) \\ 0 & \text{if } x = 0 \\ x^2 & \text{if } x \in (0, \infty) \end{cases}.$$

Is f continuous at $x = 0$?

Definition. A function f has a *removable discontinuity* at $x = a$ if $f(a)$ can be redefined in such a way that f is continuous at a. f has a *jump discontinuity* at $x = a$ if $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ exist (as finite numbers) and are different.

Example. Discuss the discontinuities of $f(x) = \dfrac{|x|}{x}$ and $g(x) = \text{int } x$.

Theorem 8. Properties of Continuous Functions

If the functions f and g are continuous at $x = c$, then the following combinations are continuous at $x = c$.

1. *Sums*: $f + g$

2. *Differences*: $f - g$

3. *Products*: $f \cdot g$

4. *Constant Multiples*: $k \cdot f$, for any number k

5. *Quotients*: f/g, provided $g(c) \neq 0$.

Theorem 9. Composite of Continuous Functions

If f is continuous at c and g is continuous at $f(c)$, then the composite $g \circ f$ is continuous at c.

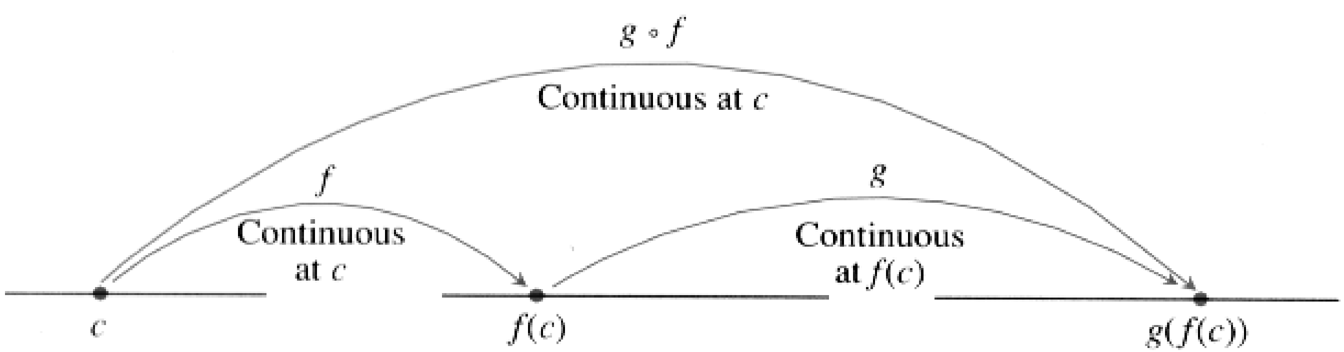

Figure 1.4.53, page 129.

Example. Page 132 number 22.

Theorem. The Intermediate Value Theorem for Continuous Functions

A function $y = f(x)$ that is continuous on a closed interval $[a, b]$ takes on every value between $f(a)$ and $f(b)$. In other words, if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.

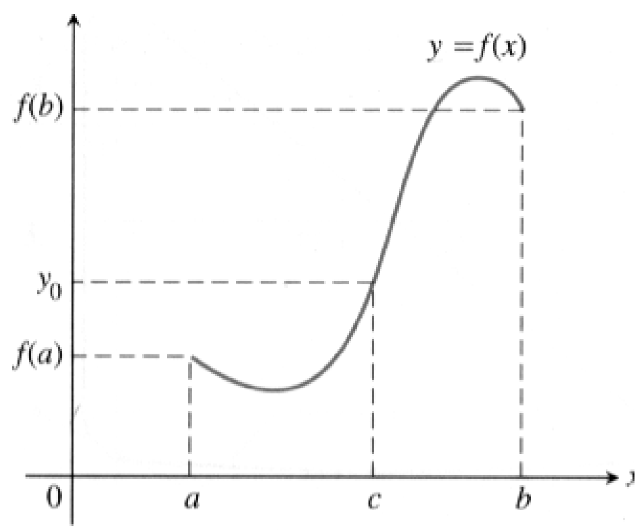

Figure 1.4.44, page 93 of 9th edition.

Example. Page 133 number 28.